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Lee waves in a stratified flow
Part 1. Thin barrier

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The lee-wave amplitudes and wave drag for a thin barrier in a two-dimensional stratified flow in which the upstream dynamic pressure and density gradient are constant (Long's model) are determined as functions of barrier height and Froude number for a channel of finite height and for a half-space. Variational approximations to these quantities are obtained and validated by comparison with the earlier results of Drazin & Moore (1967) for the channel and with the results of an exact solution for the half-space, as obtained through separation of variables. An approximate solution of the integral equation for the channel also is obtained through a reduction to a singular integral equation of potential theory. The wave drag tends to increase with decreasing wind speed, but it seems likely that the flow is unstable in the region of high drag. The maximum attainable drag coefficient consistent with stable lee-wave formation appears to be roughly two and almost certainly less than three.

1. Introduction

We consider the excitation of two-dimensional lee waves in a stratified flow over a thin, vertical barrier on the basis of Long's (1953, 1955) model, in which the dynamic pressure and the vertical density gradient in the basic flow are constant. This problem has been considered previously by Drazin & Moore (1967), who obtained an essentially numerical solution for a barrier in a channel of finite height.‡ It appears to be the only known solution of the lee-wave problem for a prescribed barrier of non-small height (linearized theory, in which the boundary condition on the barrier is applied at the ground plane, is applicable for sufficiently low barriers; see Yih 1965). We reconsider Drazin & Moore's problem in order to develop analytical approximations to the lee-wave amplitudes and the barrier drag as functions of the dimensionless parameters

$$d = \pi h/H \tag{1.1}$$

and

$$F \equiv 1/k = \pi U/NH \quad (K < k < K+1), \tag{1.2}$$

where h is the barrier height, H the channel height, N the *intrinsic* (Väisälä) *frequency* associated with the stratification, U the wind speed, F a Froude

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‡ Jones (1967) has obtained a formally exact solution of Drazin & Moore's problem for $d = \frac{1}{2}\pi$, in which special case the Wiener–Hopf technique may be applied. He also considered the problem for a step (discontinuous change in height) in a finite channel.

number, k a *reduced frequency*, and K a non-negative integer. The lee-wave spectrum of the channel contains K discrete modes. The range of physical interest appears to be roughly $k = (0{\cdot}5, 5)$, but the disturbed flow may be unstable for sufficiently large kd (see below).

We also present a solution for a half-space, a model that would appear to be appropriate for barrier heights that are small compared with the scale-height of the atmosphere and avoids the rather artificial boundary condition of a rigid upper boundary. An appropriate parameter for this limiting case, as well as for the qualitative discussion of the finite channel, is

$$\kappa = kd = Nh/U. \tag{1.3}$$

The lee-wave spectrum for the half-space is continuous (this is a consequence of the assumed distributions of density and shear, as well as the geometry). The range of physical interest appears to be roughly $\kappa = (0, 1{\cdot}5)$.

One of the striking features of stratified flow over a barrier, at least for an aerodynamicist, is the very large wave drag that is predicted by the theoretical models for non-small κ. The meteorological implications of this wave drag have been considered by Sawyer (1959) and Blumen (1965) on the basis of linearized theory. Drazin & Moore obtained, but did not comment on, drag coefficients of several thousand for a thin barrier. The following extension of their calculations reveals that the drag coefficient for the thin barrier in a finite channel is an increasing function of the Froude number $1/k$, and hence also of the wind speed, for k in $(K, K+1)$, but exhibits discontinuous decreases (for increasing $1/k$) at integral values of k; in particular, $C_D = 0$ for $k < 1$. This behaviour reflects the fact that the flow for k in $(K, K+1)$ is subcritical with respect to the first K lee-wave modes and supercritical with respect to all higher modes. The corresponding phenomenon for a barrier in a half-space, with its continuous spectrum of lee-wave modes, is also anomalous: we find that the drag is a monotonically increasing function of κ, and hence a monotonically decreasing function of wind speed, in the range considered.

It is probable that phenomena not properly described by Long's model are dominant for large κ and prevent the attainment of the larger values of C_D. Long's (1955) investigation suggests that the flow is likely to be unstable for sufficiently large obstacles if $k > 1$; the appropriate measure of 'sufficiently large' in the present investigation appears to be κ, and our results suggest that the critical value of κ is roughly $1{\cdot}5$. The model of a thin plate also is open to the objection that the real flow over such an obstacle would separate, at least locally (the turbulent wake may collapse at a distance of the order of U/N downstream from the barrier; cf. Schooley & Stewart 1963). This suggests that the local features of the predicted flow may not be realistic. It remains possible, nevertheless, that the predicted, downstream lee-wave pattern and wave drag may closely resemble those which would be observed in a real fluid in some non-trivial range of κ. In any event, it seems desirable to obtain solutions of the lee-wave problem for a few barriers of prescribed, albeit artificial, shape in order to assess the validity of approximate solutions for barriers of more realistic shapes.

The analytical techniques that we invoke have their origins in electromagnetic diffraction theory, although they have been previously applied to such fluid-mechanical problems as slender-wing theory (Miles 1959) and surface-wave scattering by a step (Miles 1967). Perhaps the most important of these is Schwinger's variational formulation, which we apply in §4 below to obtain relatively simple approximations to the lee-wave amplitudes. These approximations appear to be adequate for the parametric range of interest (the approximation for the channel is not uniformly valid for $k \to K+1-$, but neither is the basic model).

We also consider, in §5 below, an approximate solution to the integral equation for a finite channel. This formulation, which is based on a transformation of the original integral equation to a singular integral equation of the type that arises in thin-airfoil theory, leads to an infinite set of algebraic equations that must be solved by truncation. We find that $K+1$ equations yield an adequate approximation to the amplitudes of the K lee waves. Drazin & Moore's formulation also leads to an infinite set of algebraic equations, of which they solved 150 on a high-speed computer. Our formulation is far more efficient in principle, but at the expense of analytically more complicated coefficients, such that it is not useful for large K.

The boundary-value problem for a thin barrier in a half-space can be solved exactly by separation of variables, as in the well-known problem of diffraction by a plane ribbon. This solution, which we obtain in §6 below, culminates in an infinite series of Mathieu functions that converges quite rapidly for $\kappa < 6$ and gives a firm basis of comparison for the variational approximation. A similar formulation is possible for a semi-elliptical obstacle, but the expansion coefficients are coupled and can be determined only approximately by truncation of the resulting, infinite set of algebraic equations.†

2. Long's model

The basic assumptions for Long's model can be posed in the form

$$\rho(y)\,U^2(y) = 2q \tag{2.1}$$

and

$$U(y)/N(y)\,l = F \equiv 1/k, \quad N(y) = \{-g\rho'(y)/l\rho(y)\}^{\frac{1}{2}}, \tag{2.2}$$

where $\rho(y)$ and $U(y)$ are the density and wind speed in the undisturbed flow, ly is the elevation, l is a characteristic length, q is a constant dynamic pressure, and F is the Froude number based on l. [The corresponding Richardson number, $Ri \equiv (N/U')^2$, is proportional to F^2/M^4, where M is the Mach number if l is the scale height of the atmosphere; $M \ll 1$ by hypothesis, so $R_i \gg 1$ for the *undisturbed* flow.] The velocity, density, and pressure fields can then be expressed in terms of the vertical displacement of a streamline, say $l\delta(x,y)$ relative to its position in the undisturbed flow, according to

$$u = U(y-\delta)\{1-\delta_y\}, \quad v = U(y-\delta)\,\delta_x, \quad \rho = \rho(y-\delta), \tag{2.3a,b,c}$$

and

$$p = p_0 - q(\delta_x^2 + \delta_y^2 + k^2\delta^2 - 2\delta_y), \tag{2.4}$$

† I hope to present the results for the special case of a semi-circle in a sequel (part 2) to the present paper.

where subscripts denote partial differentiation with respect to the dimensionless Cartesian co-ordinates x and y, $p_0 = p_0(y)$ is the pressure in the basic flow, and δ satisfies the Helmholtz equation

$$\nabla^2\delta + k^2\delta = 0. \tag{2.5}$$

The drag on the barrier is given by the momentum integral

$$D = -l\int_C \{(p-p_0+\rho u^2)\,dy - \rho uv\,dx\}, \tag{2.6}$$

where C is a contour that encloses the barrier, and the positive sense is counter-clockwise. Substituting (2.3) and (2.4) into (2.6) and invoking conservation of mass, we obtain

$$D = ql\int_C \{(\delta_x^2 - \delta_y^2 + k^2\delta^2)\,dy - 2\delta_x\delta_y\,dx\}. \tag{2.7}$$

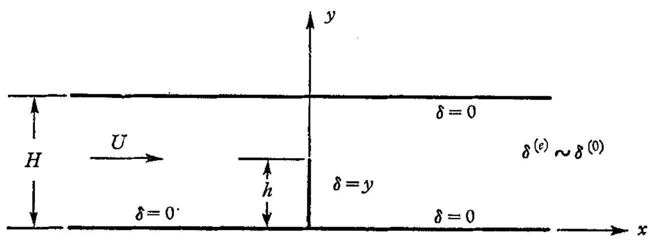

FIGURE 1. Thin barrier in channel of finite height.

The boundary-conditions for a thin barrier of height $h \equiv dl$ in a channel of height $H \equiv \pi l$ (figure 1) are

$$\delta(0,y) = y \quad (0 \leqslant y \leqslant d) \tag{2.8}$$

and

$$\delta(x,0) = \delta(x,\pi) = 0. \tag{2.9}$$

In addition, we invoke the requirement of no upstream $(x \to -\infty)$ reflexion (see below) and require δ to be bounded and continuous in the physical domain. As a particular consequence of this last requirement, we invoke the edge condition

$$|\nabla\delta| = O[\{x^2+(y-d)^2\}^{-\frac{1}{4}}] \quad (x\to 0, y \to d) \tag{2.10}$$

in order to rule out eigensolutions with physically unacceptable singularities (cf. Rayleigh 1897; Van Dyke 1964, p. 53). We call attention to the analogy with the leading-edge condition that must be invoked in thin-airfoil theory and anticipate (see §5 below) that it also may be necessary to impose a smoothness condition at the stagnation point $(x = y = 0)$ that is analogous to the Kutta condition at the trailing edge of an airfoil.

We find it expedient, in discussing the boundary conditions at infinity, to resolve $\delta(x,y)$ into odd and even functions of x, say $\delta^{(e)}(x,y)$ and $\delta^{(0)}(x,y)$, such that

$$\delta(x,y) = \delta^{(e)}(|x|,y) + \delta^{(0)}(|x|,y)\operatorname{sgn} x, \tag{2.11}$$

where both $\delta^{(e)}$ and $\delta^{(0)}$ satisfy (2.5) and (2.9). Invoking the requirements that δ

be continuous across $x = 0$ in $y = [0, \pi]$ and that δ_x be continuous across $x = 0$ in $y = (d, \pi)$, we obtain

$$\delta^{(0)}(0, y) = 0 \quad (0 \leqslant y \leqslant \pi) \tag{2.12}$$

and

$$\delta^{(e)}(0, y) = y \quad (0 \leqslant y \leqslant d), \quad \delta_x^{(e)}(0, y) = 0 \quad (d < y \leqslant \pi). \tag{2.13a, b}$$

We remark that (2.10) permits $\delta_x^{(e)}$ to be singular like

$$\delta_x^{(e)}(0, y) = O(d-y)^{-\frac{1}{2}} \quad (y \to d-). \tag{2.13c}$$

Let B denote the boundary, and S the interior, of the semi-infinite strip $x > 0$, $0 < y < \pi$. Then $\delta^{(0)}$ satisfies the homogeneous differential equation (2.5) and the homogeneous boundary conditions (2.9) and (2.12) on B and must be bounded and continuous in S; accordingly, it must either vanish identically or be an eigensolution, namely a gravity wave (or, more generally, a set of gravity waves).

The solution $\delta^{(e)}$ satisfies inhomogeneous boundary conditions on B and therefore can be resolved into a component that satisfies these boundary conditions and may comprise gravity waves and, in addition, an eigensolution, which can comprise only gravity waves. The gravity-wave portion of the solution, if it exists, dominates the asymptotic behaviour of $\delta^{(e)}$, and the amplitudes of the eigensolutions in both $\delta^{(0)}$ and $\delta^{(e)}$ then are determined by the requirement that the gravity-wave components of $\delta^{(e)}$ and $\delta^{(0)}$ cancel one another identically as $x \to -\infty$. We infer from these considerations that

$$\delta^{(0)}(x, y) \sim \delta^{(e)}(x, y) \quad (x \to \infty) \tag{2.14}$$

and

$$\delta(x, y) \sim 2\delta^{(0)}(x, y) \quad (x \to \infty) \tag{2.15}$$

unless $\delta^{(0)}$ vanishes identically, in which case $\delta^{(e)}$ is determined uniquely by the boundary conditions on B together with the requirement that the solution be bounded.

The preceding considerations provide a heuristic basis for the existence and uniqueness of a solution to the boundary-value problem posed by (2.5), (2.8)–(2.10) and (2.14). There remains the question of stability.

Long (1955) asserts that a necessary condition for stability is

$$\delta_y \leqslant 1 \tag{2.16}$$

at every point in the flow. The essential argument is that $\delta_y > 1$ implies static instability ($\partial\rho/\partial y > 0$) by virtue of (2.3$c$) and the basic assumption $\rho'(y) < 0$; it also implies $u < 0$, and hence the existence of closed streamlines, by virtue of (2.3a). Long also points out that (2.16) is not a sufficient condition for stability owing to the possibility of dynamical (shearing) instability of a statically stable flow, but concludes that the possible differences in the parametric stability criterion are likely to be small. In fact, static stability is only typically, but not always, a necessary condition for the dynamic stability of finite-amplitude disturbances, so that Long's assertion of the necessity of (2.16) must be regarded as plausible rather than certain. We adopt the point of view that its violation by the lee-wave field (2.15) casts serious doubt on the physical significance of that field. We shall not consider its relevance for the immediate neighbourhood of the barrier, where the neglect of viscosity in the basic model already casts doubt on the physical significance of the local field.

3. Barrier in finite channel

We now consider the solution of the boundary-value problem posed by (2.5), (2.8)–(2.10) and (2.14). Pursuing the analogy with thin-airfoil theory, we regard the barrier ($x = 0$, $0 < y < d$) as a vortex sheet of strength

$$g(y) = \tfrac{1}{2}\{\delta_x(0-,y)-\delta_x(0+,y)\} \tag{3.1a}$$

$$= -\delta_x^{(e)}(0+,y). \tag{3.1b}$$

Introducing the Fourier representation

$$g(y) = \sum_1^\infty G_n \sin ny, \tag{3.2}$$

where
$$G_n = (2/\pi)\int_0^d g(\eta)\sin n\eta\, d\eta, \tag{3.3}$$

and posing a similar expansion for $\delta^{(e)}$, we find that the most general solution of (2.5) that is even in x, satisfies the boundary conditions (2.9), (2.13*b*) and (3.1*b*), and is bounded and continuous in S, is given by

$$\delta^{(e)}(x,y) = \sum_1^K \{C_n \cos(k_n x) - k_n^{-1}G_n \sin(k_n|x|)\}\sin ny + \sum_{K+1}^\infty \alpha_n^{-1}G_n e^{-\alpha_n|x|}\sin ny, \tag{3.4}$$

where
$$k_n = (k^2-n^2)^{\frac{1}{2}}, \quad \alpha_n = (n^2-k^2)^{\frac{1}{2}} \quad (K < k < K+1), \tag{3.5}$$

K is the integral part of k, and the C_n are the (as yet) undetermined amplitudes of the gravity waves that make up the even eigensolution. Similarly, invoking (2.12) in place of (3.1*b*), we obtain the odd eigensolution

$$\delta^{(0)}(x,y) = \sum_1^K S_n \sin(k_n x)\sin ny. \tag{3.6}$$

Invoking (2.14), we obtain

$$C_n = 0, \quad S_n = -k_n^{-1}G_n \quad (n = 1,\ldots,K). \tag{3.7a,b}$$

Substituting (3.3) and (3.7*a*) into (3.4) and invoking (2.13*a*, *b*), we obtain

$$\int_0^d K(y,\eta)g(\eta)\,d\eta = y \quad (0 < y < d) \tag{3.8a}$$

and
$$g(y) = 0 \quad (d < y < \pi), \tag{3.8b}$$

where
$$K(y,\eta) = (2/\pi)\sum_{K+1}^\infty \alpha_n^{-1}\sin ny \sin n\eta. \tag{3.9}$$

We note that the above Fourier series cannot be expected to converge uniformly in the neighbourhoods of any of $y = 0, d$ and π.

The preceding formulation is equivalent to that of Drazin & Moore after allowing for the following changes of notation: $T = H$, $\lambda_n = |k_n| = |\alpha_n|$ and $\lambda_n A_n = G_n$. We anticipate that the G_n remain bounded at both $k \to K+$ and $k \to K+1-$ and therefore provide a more satisfactory basis for calculation than do the S_n (since $S_K \to \infty$ as $k \to K+$).

Substituting

$$\delta \sim -2\sum_{1}^{K} k_n^{-1} G_n \sin(k_n x) \sin ny \tag{3.10}$$

into (2.7) after choosing C as the rectangle $y = 0, \pi, x = -\infty, +\infty$, we obtain the barrier drag coefficient

$$C_D = (qh)^{-1} D = (2\pi/d) \sum_{1}^{K} G_n^2, \tag{3.11}$$

which vanishes identically for $k < 1$ and exhibits finite discontinuities at $k = 1, 2, \ldots$ (see §§4 and 5 below). Invoking (2.16) for the lee-wave field, we obtain

$$\max\left\{-2\sum_{1}^{K} nk_n^{-1} G_n \sin(k_n x) \cos ny\right\} < 1 \tag{3.12}$$

as the necessary condition for static stability. If $K = 1$, (3.12) implies

$$G_1 < \tfrac{1}{2}(k^2-1)^{\frac{1}{2}}, \quad C_D < \tfrac{1}{2}\pi d^{-1}(k^2-1) \quad (K = 1). \tag{3.13a, b}$$

Explicit solutions of (3.12) are not possible for $K > 1$; however, a fair approximation, and certainly a necessary condition, is given by

$$G_K < \tfrac{1}{2}K^{-1}(k^2 - K^2)^{\frac{1}{2}}. \tag{3.14}$$

Arbitrary obstacle

The preceding development provides the Green's function for an obstacle of arbitrary shape in a channel of finite height. Let $G(x, y, \xi, \eta)$ satisfy

$$\nabla^2 G + k^2 G = -\delta_1(x-\xi)\,\delta_1(y-\eta), \tag{3.15}$$

$$G = 0 \quad (y = 0, \pi), \tag{3.16}$$

and

$$G \to 0 \quad (x \to -\infty), \tag{3.17}$$

where δ_1 is Dirac's delta function. Applying Green's second theorem to $\delta(x, y)$ and G around a closed contour made up of the obstacle, $y = 0$ outside of the obstacle, $y = \pi$ and $x = \pm\infty$, we obtain

$$\delta(x, y) = \int_{\text{obstacle}} \left\{\frac{\partial G}{\partial n}\,\delta(\xi, \eta) - G\frac{\partial}{\partial n}\,\delta(\xi, \eta)\right\} dl, \tag{3.18}$$

where n is the outwardly directed normal to the obstacle. We infer from the preceding development that the solution to (3.15)–(3.17) is given by

$$\pi G(x, y, \xi, \eta) = \left\{-2H(x-\xi)\sum_{1}^{K} k_n^{-1} \sin k_n(x-\xi) + \sum_{K+1}^{\infty} \alpha_n^{-1} e^{-\alpha_n |x-\xi|}\right\} \sin ny \sin n\eta, \tag{3.19}$$

where H is Heaviside's step function. Substituting (3.19) into (3.18), choosing (x, y) on the obstacle, and setting $\delta(x, y) = y$ and $\delta(\xi, \eta) = \eta$, we obtain an integral equation for the determination of $\partial\delta/\partial n$ on the obstacle. The solution of this integral equation would permit the determination of $\delta(x, y)$ from (3.18).

4. Variational approximation

Let $g^*(y)$ be a trial function that is continuous in $y = (0, d)$, satisfies (2.13c), and vanishes identically in $y = (d, \pi)$. Substituting

$$g(y) = Cg^*(y) \tag{4.1}$$

into (3.8a), multiplying both sides of the result by $g^*(y)$, and integrating over $y = (0, d)$, we obtain

$$C = \int_0^d yg^*(y)\,dy \Big/ \int_0^d \int_0^d g^*(y) K(y,\eta) g^*(\eta)\,d\eta\,dy \tag{4.2a}$$

$$= (2/\pi)\int_0^d yg^*(y)\,dy \Big/ \left\{\sum_{K+1}^{\infty} a_n^{-1}(G_n^*)^2\right\}, \tag{4.2b}$$

where
$$G_n^* = (2/\pi)\int_0^d g^*(y)\sin ny\,dy. \tag{4.3}$$

The corresponding approximation to G_n is

$$G_n = CG_n^*. \tag{4.4}$$

We may associate the approximation provided by (4.1)–(4.4) with the variational integral

$$I \equiv \int_0^d yg(y)\,dy = \left\{\int_0^d yg^*(y)\,dy\right\}^2 \Big/ \int_0^d \int_0^d g^*(y) K(y,\eta) g^*(\eta)\,d\eta\,dy, \tag{4.5}$$

which is stationary with respect to first-order variations of $g^*(y)$ about the true solution to (3.8a). We could use this variational principle to obtain systematic approximations to $g(y)$; however, we rest content with the direct approximation of (4.1)–(4,4) on the basis of an assumed form for $g^*(y)$. We may demonstrate, on the basis of the rather more general variational formulation developed in the appendix, that the error in the approximation (4.4) is of the order of the square of the error in the trial function g^*.

A suitable trial function for small kd is provided by the solution of Laplace's equation for a barrier in a half-space [the complex potential is $\{(x+iy)^2+d^2\}^{\frac{1}{2}}$], which yields

$$g^*(y) = y(d^2-y^2)^{-\frac{1}{2}}. \tag{4.6}$$

Substituting (4.6) into (4.3)–(4.4), we obtain

$$G_n = CdJ_1(nd), \quad C = \left\{2\sum_{K+1}^{\infty} \alpha_n^{-1} J_1^2(nd)\right\}^{-1}, \tag{4.7a,b}$$

where J_1 is a Bessel function. The corresponding approximation to C_D, as determined from (3.11), is

$$C_D = 2\pi C^2 d \sum_1^K J_1^2(nd). \tag{4.8}$$

We observe that $C \to 0$ as $k \to K+1-$ (the results of the following section reveal that G_n and C_D actually have finite values at $k = K+1-$) and then jump discontinuously to positive values as k increases through $K+1$.

The approximations provided by (4.7) and (4.8) for $d = \frac{1}{4}\pi$ are plotted in figures 2 and 3. The criterion (3.12) is violated in $1 < k < 1{\cdot}24$ for $K = 1$, in $2 < k < 2{\cdot}85$ for $K = 2$, and throughout almost the entire range $3 < k < 4$ for $K = 3$. The physical significance of the results in these ranges is dubious, but

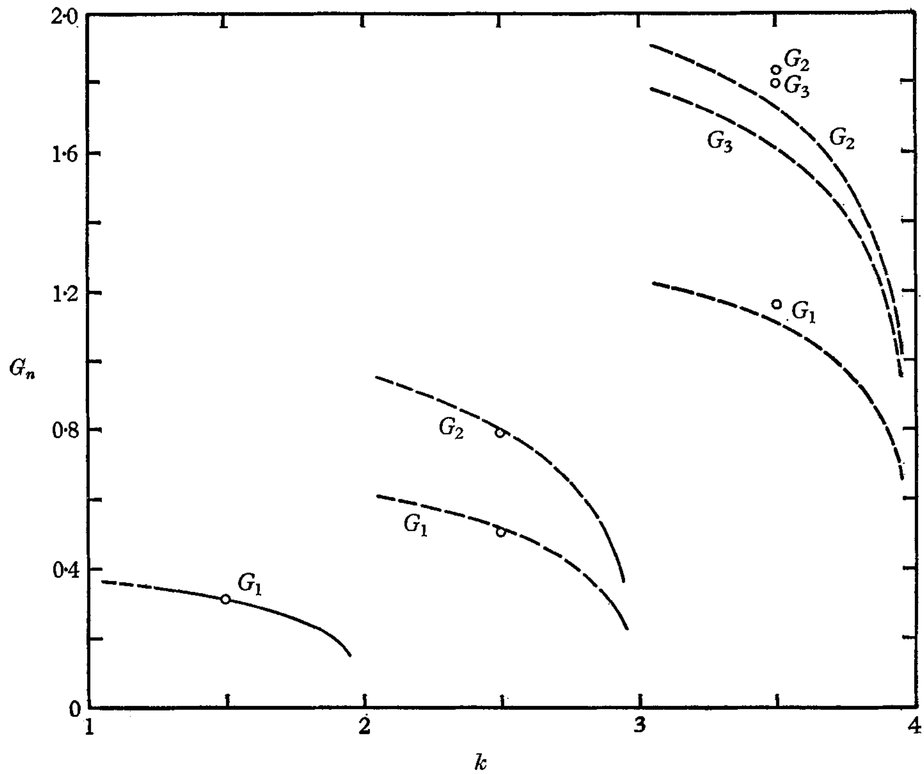

FIGURE 2. Variational approximations to $G_1, \ldots, G_K$ for $d = \frac{1}{4}\pi$, as given by (4.7). The circles denote values calculated by Drazin & Moore. The stability criterion (3.12) is violated over the dashed portions of the curves.

we have included them for numerical comparison with the corresponding points calculated by Drazin & Moore. We infer from this comparison that the error in G_n at $k = K + \frac{1}{2}$ and $d = \frac{1}{4}\pi$ is roughly $10^{K-2}\,\%$. The approximation provided by (4.7) for $K = 1$ and $d = \frac{1}{2}\pi$ is plotted in figure 4. It differs from Jones's (1967) result by approximately $2\frac{1}{2}\,\%$ at $k = 1{\cdot}5$; however, (3.12) is violated throughout $1 < k < 2$. The variations of G_n and C_D with d for fixed k are shown in figures 5 and 6 (the variational approximation appears to be less accurate than that of §5 below for $d/\pi > 0{\cdot}4$, and figures 5 and 6 actually are based on the results obtained there; the differences between the two approximations are less than $1\,\%$ for $d/\pi < 0{\cdot}4$). The maximum values of κ consistent with laminar flow appear to lie between 1 and 2 [the critical values of κ calculated by Long (1955) for obstacles that approximate truncated sine waves are roughly unity]. We surmise that drag coefficients substantially greater than two or three are not likely to be attainable in laminar flows.

The limiting forms of (4.7) and (4.8) as $d \to 0$ with k fixed are

$$G_n \to \tfrac{1}{2}nd^2 \quad (d \to 0) \tag{4.9}$$

and

$$C_D \to \tfrac{1}{6}\pi K(K+\tfrac{1}{2})(K+1)d^3 \quad (d \to 0). \tag{4.10}$$

The result (4.9) yields a solution equivalent to that given by Drazin & Moore for a dipole of strength $\mu = \frac{1}{2}\pi d^2$ or, equivalently, a small, semicircular barrier of

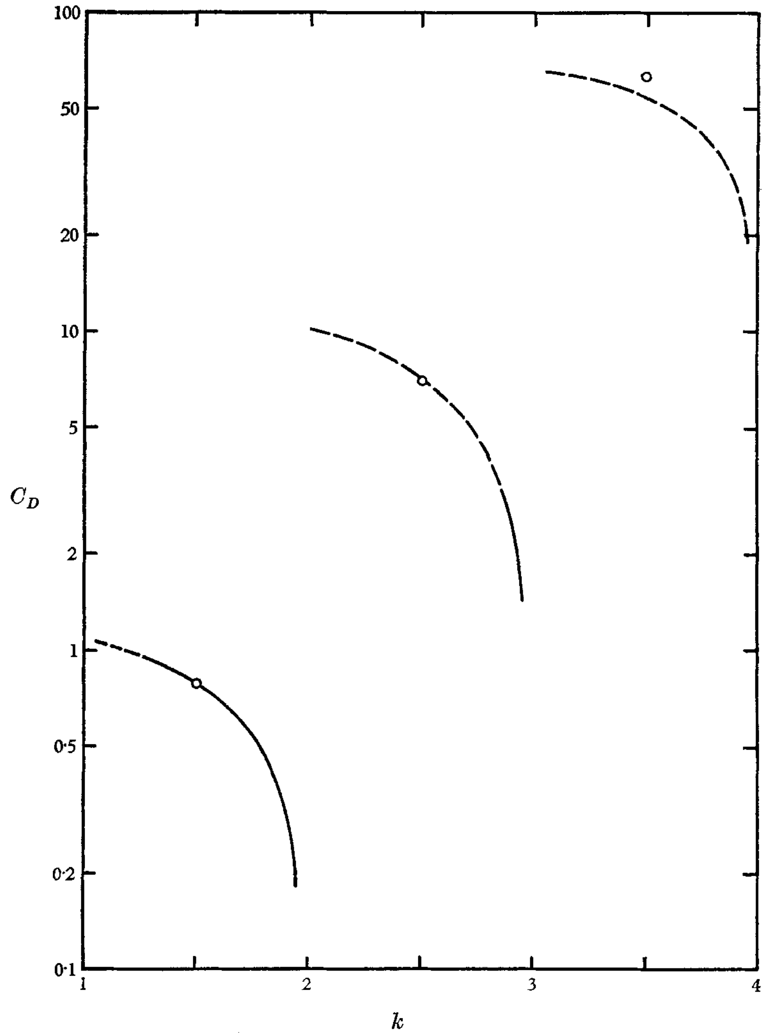

FIGURE 3. Variational approximation to C_D, as given by (4.8) for $d = \frac{1}{4}\pi$. The circles denote values calculated by Drazin & Moore. The stability criterion (3.12) is violated over the dashed portions of the curves.

height $h/\sqrt{2}$. There is, however, a numerical discrepancy in the corresponding result for C_D, which appears to be associated with an error in Drazin & Moore's calculation. We find that their result (5.8) for the drag on a dipole should be multiplied by $(4/\pi^4)$ and that their result for the drag on a small, semicircular barrier of radius A should be multiplied by 4.

The limiting forms of (4.7 *b*) and (4.8) as $k \to \infty$ with kd fixed, corresponding to a thin barrier in a half space (figure 7), are

$$C = \left\{2\int_{\kappa}^{\infty} (\nu^2-\kappa^2)^{-\frac{1}{2}} J_1^2(\nu)\,d\nu\right\}^{-1} \tag{4.11}$$

and

$$C_D = 2\pi C^2 \int_0^{\kappa} J_1^2(\nu)\,d\nu, \tag{4.12}$$

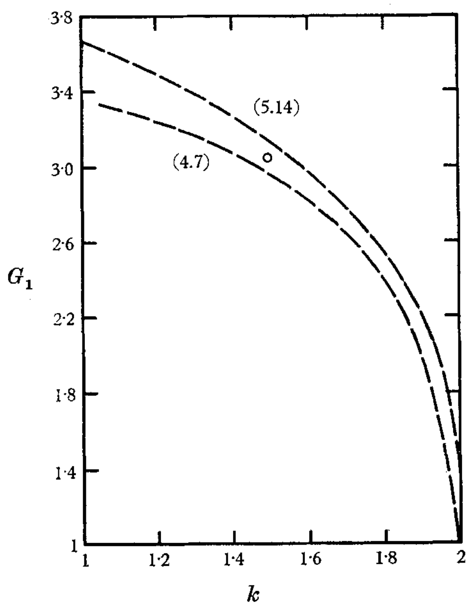

FIGURE 4. G_1 for $K = 1$ and $d = \frac{1}{2}\pi$, as given by the variational approximation (4.7) and the Galerkin approximation (5.14). The circle denotes a value calculated by Jones (1967). The stability criterion (3.12) is violated over the entire range $1 < k < 2$.

where $\kappa = kd$ is defined as in (1.3). Introducing Neumann's integral representation for $J_1^2(\nu)$ in (4.11) and evaluating the resulting integral with respect to ν, we obtain the alternative representation

$$C = \left\{-2\int_0^{\kappa} (\kappa^2-\nu^2)^{-\frac{1}{2}} J_1(\nu)Y_1(\nu)\,d\nu\right\}^{-1}, \tag{4.13}$$

which is preferable to (4.11) for numerical evaluation. Introducing the series representations for J_1^2 and J_1Y_1, we obtain

$$C^{-1} = 1-(\tfrac{1}{2}\kappa)^2(\log\tfrac{1}{4}\kappa+\gamma+\tfrac{1}{4})+\tfrac{3}{4}(\tfrac{1}{2}\kappa)^4(\log\tfrac{1}{4}\kappa+\gamma-\tfrac{1}{4})+O\{(\tfrac{1}{2}\kappa)^6\log\kappa\} \tag{4.14}$$

and

$$C_D = \tfrac{1}{6}\pi\kappa^3C^2\{1-\tfrac{3}{5}(\tfrac{1}{2}\kappa)^2+\tfrac{5}{28}(\tfrac{1}{2}\kappa)^4+O(\tfrac{1}{2}\kappa)^6\} \tag{4.15a}$$

$$= \tfrac{1}{6}\pi\kappa^3\{1+\tfrac{1}{2}\kappa^2(\log\tfrac{1}{4}\kappa+\gamma-\tfrac{1}{20})+O(\kappa^4\log^2\kappa)\}. \tag{4.15b}$$

The approximation (4.15*b*) is exact to the indicated order by virtue of the variational principle and the fact that the error in the approximation $g = g^*$ is $1+O(\kappa^2)$. The limiting drag coefficient implied by (4.15), namely $C_D = \frac{1}{6}\pi\kappa^3$, is equivalent to that given by (4.10) in the joint limit $K\to\infty$, $d \to 0$.

The approximation (4.12) is plotted in figure 8, together with the result obtained from the formally exact solution of §6 below. The two results are indistinguishable, in the scale of the drawing, for $\kappa < 2{\cdot}8$, which appears to cover the range of physical interest (see below). The restriction corresponding to (3.12) is

$$\max\left\{-2\int_0^\kappa \nu(\kappa^2-\nu^2)^{-\frac{1}{2}}J_1(\nu)\sin[(\kappa^2-\nu^2)^{\frac{1}{2}}x]\cos\nu y\,d\nu\right\} < 1. \tag{4.16}$$

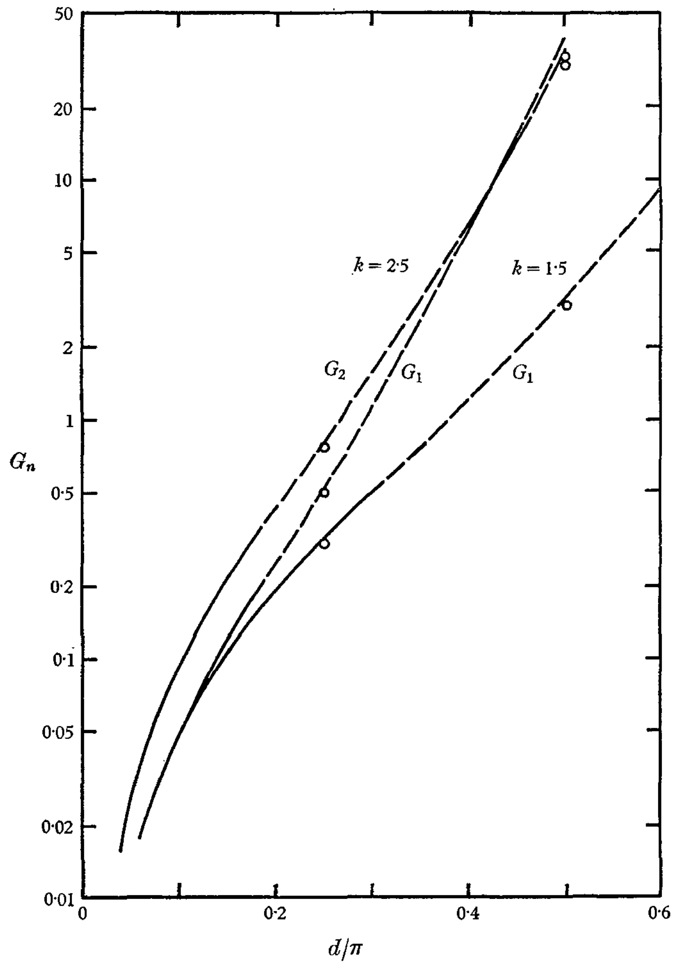

FIGURE 5. G_1 for $k = 1{\cdot}5$, as given by (5.14), and G_1 and G_2 for $k = 2{\cdot}5$, as determined by truncating (5.11) at $n = 3$. The circles denote values calculated by Drazin & Moore. The stability criterion (3.12) is violated over the dashed portions of the curves.

The integral is intractable, but we obtain an upper bound by replacing the trigonometric product by -1; the corresponding, *lower* bound to the critical value of κ is $1{\cdot}25$. Referring to figure 5, we find that the critical values of kd for the finite channel are $1{\cdot}47$ and $1{\cdot}50$ for $k = 1{\cdot}5$ and $2{\cdot}5$, respectively (our choice of these intermediate values of k tends to minimize the effects of the discontinuities at integral values of k). We surmise from these considerations that the critical value

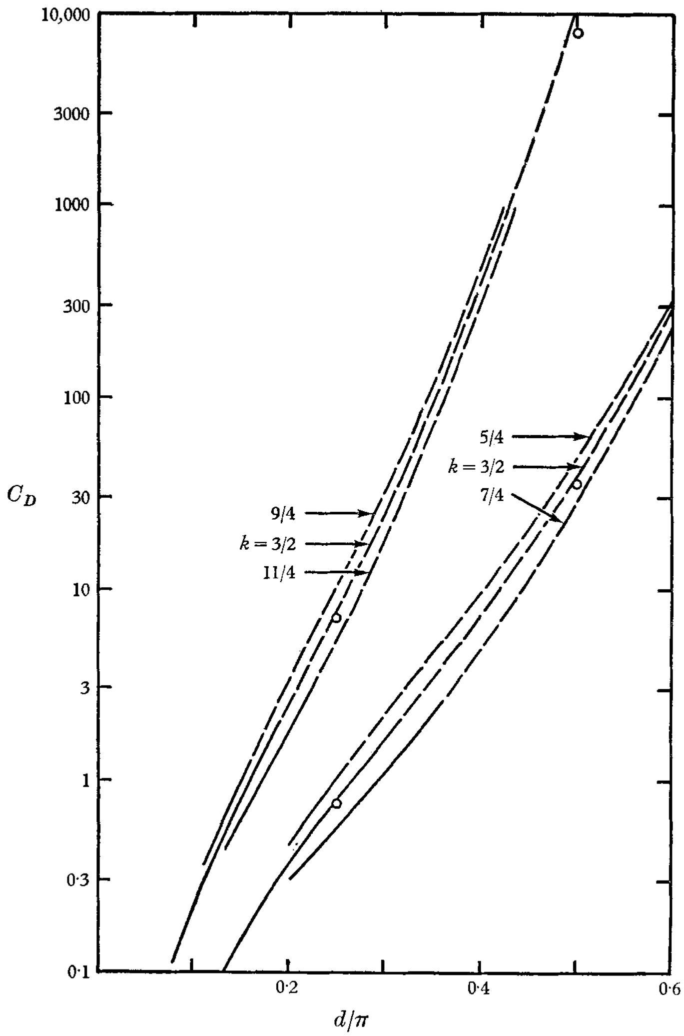

FIGURE 6. C_D *vs.* d/π, as determined by truncating (5.11) at $n = 2$ for $K = 1$ and at $n = 3$ for $K = 2$. The stability criterion (3.12) is violated over the dashed portions of the curves. The circles denote values calculated by Drazin & Moore.

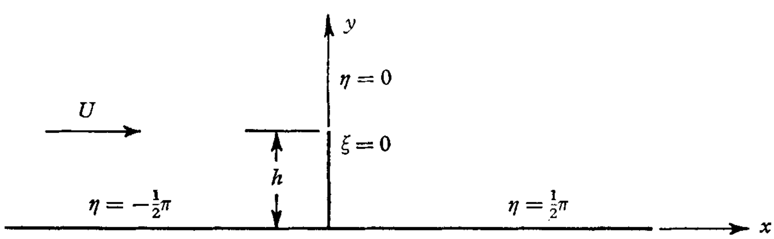

FIGURE 7. Thin barrier in half-space.

of κ for the half-space is roughly 1·5; it is almost certainly less than two. The corresponding drag coefficient is roughly two and almost certainly less than three. (See note at end of §6 below.)

We infer from these last results (together with the supporting results of §6 below) that not only C_D, but also C_D/κ^2 (to which the drag is proportional as U varies), are monotonically increasing functions of κ in the range of laminar flow. This implies the anomalous result, anticipated in §1, that the drag decreases monotonically with increasing speed.

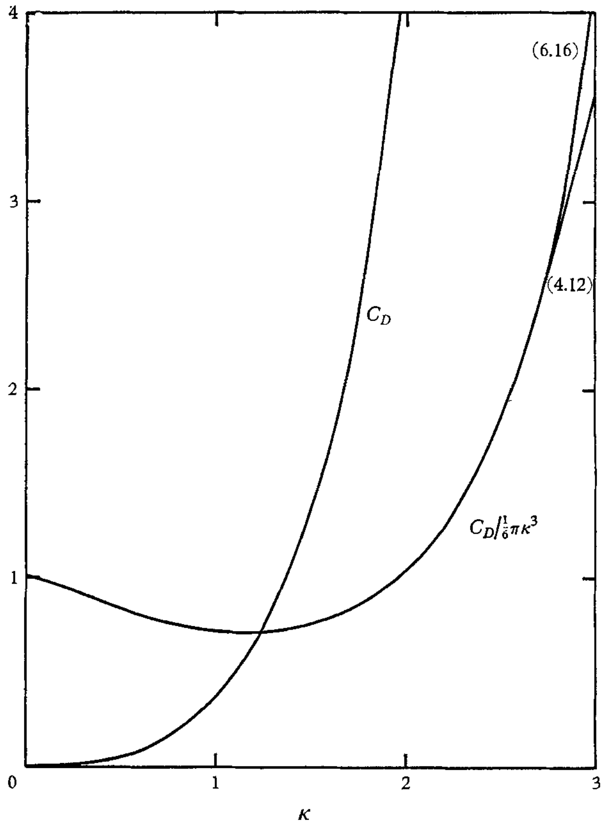

FIGURE 8. C_D *vs.* κ for the barrier of figure 7, as determined by the variational approximation (4.12) and by (6.16). The two results differ only for $\kappa > 2{\cdot}8$, where (4.12) underestimates the theoretical drag. The associated flow is likely to be unstable for $\kappa > 1{\cdot}7$.

5. Reduction of integral equation

We now obtain an approximate solution to the integral equation (3.8a) by transforming it to a corresponding integral equation of potential theory [cf. the 'equivalent-static method' of wave-guide theory (Marcuvitz 1951, p. 153)].

We begin by separating out the static ($k = 0$) portion of (3.9) according to

$$K(y,\eta) = K_0(y,\eta) - (2/\pi)\sum_1^\infty n^{-1}\epsilon_n \sin ny \sin n\eta, \tag{5.1}$$

where
$$K_0(y,\eta) = (2/\pi)\sum_1^\infty n^{-1}\sin ny \sin n\eta \tag{5.2a}$$
$$= \frac{1}{\pi}\log\left|\frac{\sin\frac{1}{2}(y+\eta)}{\sin\frac{1}{2}(y-\eta)}\right|, \tag{5.2b}$$
and
$$\epsilon_n = 1 \quad (n = 1,\ldots,K) \tag{5.3a}$$
$$= 1-n\alpha_n^{-1} = 1-\{1-(k/n)^2\}^{-\frac{1}{2}} \quad (n \geqslant K+1). \tag{5.3b}$$

Substituting (5.1) into (3.8a) and differentiating the result with respect to y, we obtain
$$\frac{1}{\pi}\int_0^d \frac{g(\eta)\sin\eta\, d\eta}{\cos y-\cos\eta} = f(y) \quad (0 < y < d), \tag{5.4}$$
where
$$f(y) = 1+\sum_1^\infty \epsilon_n G_n \cos ny, \tag{5.5}$$

and, here and subsequently, the Cauchy principal values of improper integrals are implied. Anticipating that (5.4) may yield solutions that are singular at the end points ($y = 0, d$) in consequence of the preceding differentiation, we add to (2.13c) the additional requirement that $g(y)$ be bounded at $y = 0$ (in fact, it must vanish there); accordingly

$$g(y) = O(1) \quad (y \to 0) \tag{5.6a}$$
and
$$g(y) = O(d-y)^{-\frac{1}{2}} \quad (y \to d-). \tag{5.6b}$$

The singular integral equation (5.4), which arises in both thin-airfoil and slender-wing theories, as well as in other branches of mathematical physics, can be inverted with the aid of the joint transformation

$$\cos y = a\cos\theta+1-a, \quad \cos\eta = a\cos\phi+1-a, \quad a = \sin^2\tfrac{1}{2}d \tag{5.7}$$

and the expansion
$$g(y) = \csc\theta \sum_{s=0}^\infty \mathscr{G}_s \cos s\theta. \tag{5.8}$$
The leading term in (5.8),
$$\mathscr{G}_0 \csc\theta = \mathscr{G}_0 a(1-\cos y)^{-\frac{1}{2}}(\cos y-\cos d)^{-\frac{1}{2}}, \tag{5.9}$$
is a singular eigensolution of (5.4), and $\mathscr{G}_0$ must be determined by the auxiliary condition (5.6a), which now appears as the analogue of the Kutta condition in airfoil theory. The required inversion is (cf. Soehngen 1939)

$$g(y) = \frac{1}{\pi}\tan\tfrac{1}{2}\theta \int_0^\pi \frac{f(\eta)(1+\cos\phi)\,d\phi}{\cos\phi-\cos\theta} \tag{5.10a}$$
$$= \frac{1}{\pi}\left(\frac{1-\cos y}{\cos y-\cos d}\right)^{\frac{1}{2}}\int_0^d \left(\frac{\cos\eta-\cos d}{1-\cos\eta}\right)^{\frac{1}{2}} \frac{f(\eta)\sin\eta\,d\eta}{\cos\eta-\cos y}. \tag{5.10b}$$

Substituting (5.5) into (5.10a), multiplying both sides of the result by $(2/\pi m)\sin my$, and integrating over $y = (0,d)$, we obtain

$$m^{-1}G_m - \sum_1^\infty \epsilon_n I_{mn} G_n = I_{m0} \quad (m = 1,2,\ldots), \tag{5.11}$$

where
$$I_{mn} = I_{nm} = \frac{2a}{m\pi^2}\int_0^\pi \frac{\sin my}{\sin y}(1-\cos\theta)\,d\theta \int_0^\pi \frac{(1+\cos\phi)\cos n\eta\, d\phi}{\cos\phi-\cos\theta}. \tag{5.12}$$

We note the particular values

$$\left.\begin{aligned}&I_{10} = 2a, \quad I_{20} = 2a(1-\tfrac{3}{2}a), \quad I_{30} = 2a(1-4a+\tfrac{10}{3}a^2),\\ &I_{11} = 1-(1-a)^2, \quad I_{12} = I_{21} = 2a(1-a)^2,\\ &I_{13} = I_{31} = a(1-a)^2(2-5a), \quad I_{22} = \tfrac{1}{2}-\tfrac{1}{2}(1-a)^4-4a^2(1-a)^2,\\ &I_{23} = I_{32} = 2a(1-a)^2(1-4a+6a^2),\\ &I_{33} = \tfrac{1}{3}-\tfrac{1}{3}(1-a)^6-a^2(1-a)^2(12-36a+33a^2).\end{aligned}\right\} \tag{5.13}$$

We may obtain approximate solutions to (5.11) by truncation, anticipating that convergence will be much more rapid than in Drazin & Moore's application of Galerkin's method by virtue of our prior separation of the limiting result for $k = 0$. Truncating at $n = 2$ with $K = 1$, we obtain

$$G_1 = \frac{2a}{(1-a)^2} - \frac{4a^2(2+a)\{1-(1-\frac{1}{4}k^2)^{\frac{1}{2}}\}}{1-(1-a)^4\{1-(1-\frac{1}{4}k^2)^{\frac{1}{2}}\}} + O(a^2\epsilon_3) \quad (K = 1). \tag{5.14}$$

The result (5.14) is in close agreement with the variational approximation of (4.7) except in the neighbourhood of $k = 2-$, where it correctly yields a non-zero value of G_1. The two approximations for $d = \frac{1}{2}\pi$ are compared in figure 4 (they cannot be distinguished, on the scale of figure 2, for $d = \frac{1}{4}\pi$). They appear to be of comparable accuracy at $k = 1{\cdot}5$, where each differs from Jones's (1967) result by approximately $2\frac{1}{2}$ %.

We also have obtained an approximate solution to (5.11) for $K = 2$ by truncating at $n = 3$. The results are again close to the variational approximation of (4.7) and (4.8) for $d = \frac{1}{4}\pi$, except in the neighbourhood of $k = 3-$. They are quite superior to this variational approximation for d/π substantially greater than 0·4 and were used for the computation of figures 5 and 6.

6. Barrier in half-space

We now consider a thin barrier of height $h \equiv l$ in a half-space, as shown in figure 7. This configuration is equivalent to that of §§2–5 above in the limit $H \to \infty$ with $kd = \kappa$ fixed. The formulation of §3 could be carried over by transforming the Fourier series to Fourier integrals; however, the solution through separation of variables is more direct.

Introducing elliptic co-ordinates ξ and η according to

$$x = \sinh\xi\sin\eta, \quad y = \cosh\xi\cos\eta, \tag{6.1}$$

and writing $\delta = \delta(\xi,\eta)$, rather than $\delta(x,y)$, we transform the boundary-value problem of §2 to

$$\delta_{\xi\xi} + \delta_{\eta\eta} + \kappa^2(\cosh^2\xi - \cos^2\eta)\delta = 0, \tag{6.2}$$

$$\delta^{(e)}(\xi, \pm\tfrac{1}{2}\pi) = 0, \quad \delta^{(e)}(0,\eta) = \cos\eta, \tag{6.3a,b}$$

and

$$\delta^{(0)}(\xi, \pm\tfrac{1}{2}\pi) = \delta^{(0)}(\xi,0) = \delta^{(0)}(0,\eta) = 0. \tag{6.4a,b,c}$$

Separating variables, we find that the most general even (in η) solution of (6.2) that satisfies (6.3a,b) is given by

$$\delta^{(e)}(\xi,\eta) = \sum_{n=0}^{\infty}\sum_{j=1}^{2} C^{(j)}_{2n+1} Mc^{(j)}_{2n+1}(\xi)\, ce_{2n+1}(\eta), \tag{6.5}$$

where $ce_{2n+1}(\eta)$ is an even, periodic Mathieu function, $Mc^{(j)}_{2n+1}(\xi)$ is the corresponding radial function of the jth kind, and

$$\sum_{j=1}^{2} C^{(j)}_{2n+1} Mc^{(j)}_{2n+1}(0) = (4/\pi)\int_0^{\frac{1}{2}\pi} \cos\eta\, ce_{2n+1}(\eta)\,d\eta = A_1^{(2n+1)}. \tag{6.6}$$

Our notation for the Mathieu functions is that of Abramowitz & Stegun (1964) with $q = \frac{1}{4}\kappa^2$ therein.

Similarly, the most general odd solution of (6.2) that satisfies (6.4a, b, c) is given by

$$\delta^{(0)} = \sum_1^\infty C_{2n}\, Ms^{(1)}_{2n}(\xi)\, se_{2n}(\eta), \tag{6.7}$$

where se_{2n} is an odd, periodic Mathieu function, $Ms^{(1)}_{2n}(\xi)$ is the corresponding radial function of the first kind, and C_{2n} is to be determined by the requirement (2.14).

The leading terms in the asymptotic expansions of $Mc^{(j)}_{2n+1}$ and $Ms^{(1)}_{2n}$ for $\kappa\cosh\xi \to \infty$ are given by (Abramowitz & Stegun 1964, p. 740)

$$Ms^{(1)}_{2n}(\xi) \sim -Mc^{(2)}_{2n+1}(\xi) \sim (-)^n(\tfrac{1}{2}\pi\kappa\cosh\xi)^{-\frac{1}{2}}\cos(\kappa\cosh\xi - \tfrac{1}{4}\pi) \tag{6.8}$$

and

$$Mc^{(1)}_{2n+1}(\xi) \sim (-)^n(\tfrac{1}{2}\pi\kappa\cosh\xi)^{-\frac{1}{2}}\sin(\kappa\cosh\xi - \tfrac{1}{4}\pi). \tag{6.9}$$

Substituting (6.8) and (6.9) into (6.5) and (6.7) and invoking (2.14), we obtain

$$C^{(1)}_{2n+1} = 0, \quad \sum_1^\infty (-)^n C_{2n}\, se_{2n}(\eta) = \sum_0^\infty (-)^{n-1} C^{(2)}_{2n+1}\, ce_{2n+1}(\eta). \tag{6.10a, b}$$

Invoking the orthogonality of the $se_{2n}(\eta)$, which constitute a complete set in $\eta = (0, \frac{1}{2}\pi)$, we obtain

$$C_{2m} = (-)^{m-1}(4/\pi)\sum_{n=0}^\infty (-)^n C^{(2)}_{2n+1}\int_0^{\frac{1}{2}\pi} se_{2m}(\eta)\, ce_{2n+1}(\eta)\,d\eta. \tag{6.11}$$

Substituting (6.6), (6.8) and (6.10a) into (6.5) and invoking (2.15), we obtain the asymptotic approximation to the lee wave(s) in the form

$$\delta \sim 2(\kappa r)^{-\frac{1}{2}}\cos(\kappa r - \tfrac{1}{4}\pi)F(\eta) \quad (\cosh\xi \to r \to \infty,\ 0 < \eta < \tfrac{1}{2}\pi), \tag{6.12}$$

where

$$F(\eta) = \sum_0^\infty (-)^n F_{2n+1}(\kappa)\, ce_{2n+1}(\eta), \tag{6.13}$$

$$F_n = -(2/\pi)^{\frac{1}{2}} A_1^{(n)}/Mc^{(2)}_n(0) \tag{6.14a}$$

$$= A_1^{(n)} g_{e,n}/f_{e,n}, \tag{6.15}$$

$f_{e,n}$ and $g_{e,n}$ are the *joining factors* for the radial solutions, and r is the cylindrical radius from $x = y = 0$.

Substituting (6.12) into (2.7) and choosing C as the semicircle bounded by $\eta = \mp\frac{1}{2}\pi$ and $r \to \infty$, we obtain [note that $\delta \sim o(r^{-\frac{1}{2}})$ in $\eta = (-\frac{1}{2}\pi, 0)$]

$$C_D = (qh)^{-1}D = 4\kappa\int_0^{\frac{1}{2}\pi} F^2(\eta)\sin\eta\,d\eta. \tag{6.16}$$

Using the tabulated values of $A_m^{(n)}$, $f_{e,n}$ and $g_{e,n}$, we find that the approximation

$$F(\eta) = (F_1A_1^{(1)} - F_3A_1^{(3)})\cos\eta + (F_1A_3^{(1)} - F_3A_3^{(3)})\cos 3\eta \tag{6.17}$$

yields an accuracy of about 1 % in the corresponding approximation to C_D for $\kappa < 6$ (although, as estimated in §4 above, the flow for $\kappa > 1{\cdot}5$ is likely to be unstable). We also may obtain $F(\eta)$ and C_D as expansions in κ^n and $\kappa^n \log \kappa$. The expansions of $A_1^{(n)}$ and ce_n converge quite rapidly for $\kappa < 4$, but the expansion of $Mc_n^{(2)}(0)$ converges very slowly for $\kappa > 1$, so that the results are not too useful for computation; however, they do confirm the variational approximation (4.15*b*) to the indicated order. [*Note added in proof:* Mr. H. E. Huppert has used the results of this section to establish that (2.16) is violated for $\kappa > 1{\cdot}73$. The corresponding value of C_D is 2·26.]

7. Conclusions

We conclude that the approximate methods considered in §§4 and 5 above are adequate for the calculation of lee-wave amplitudes and wave drag within the régime in which Long's model appears to be valid, namely $Nh/U < 1{\cdot}5$. We also conclude that the drag coefficient for a thin barrier in a stratified, laminar flow is not likely to exceed two or, at most, three.

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Appendix. Variational formulation

Let $g_n(y)$ be determined by

$$\int_0^d K(y,n)\,g_n(\eta)\,d\eta = \sin ny \quad (0 \leqslant y < d) \tag{A 1a}$$

and

$$g_n(y) = 0 \quad (d < y \leqslant \pi); \tag{A 1b}$$

then the required solution of (3.8*a, b*) is given by

$$g(y) = \sum_{n=1}^{\infty} Y_n g_n(y), \tag{A 2}$$

where Y_n is the finite sine transform of y, defined as in (3.2) and (3.3). We also introduce

$$G_{mn} = (2/\pi)\int_0^d g_m(y)\sin ny\,dy, \tag{A 3}$$

the finite sine transform of $g_m(y)$. Multiplying (A 1*a*) through by $(2/\pi)g_m(y)$ and integrating over $y = (0, d)$, we obtain the alternative expression

$$G_{mn} = (2/\pi)\int_0^d\int_0^d g_m(y)\,K(y,n)\,g_n(\eta)\,d\eta\,dy. \tag{A 4}$$

Dividing (A 4) through by $G_{mn}G_{nm}$, as given by (A 3), we obtain

$$\frac{1}{G_{mn}} = \frac{1}{G_{nm}} = \frac{\tfrac{1}{2}\pi\displaystyle\int_0^d\int_0^d g_m(y)\,K(y,n)\,g_n(\eta)\,d\eta\,dy}{\displaystyle\int_0^d g_m(y)\sin ny\,dy\int_0^d g_n(\eta)\sin m\eta\,d\eta}, \tag{A 5}$$

which is stationary with respect to independent, first-order variations of each of $g_m(y)$ and $g_n(y)$ about the true solutions to (A 1a), symmetric in m and n, and invariant under independent scale transformations of g_m and g_n.

The simplest variational approximation is provided by the substitution

$$g_m(y) = g_n(y) = g^*(y), \tag{A 6}$$

where $g^*(y)$ is a trial function that is continuous in $y = (0, d)$, satisfies (2.13c), and, by definition, vanishes identically in $y = (d, \pi)$. Substituting (A 6) into (A 5), calculating

$$G_m = \sum_{n=1}^{\infty} G_{mn} Y_n, \tag{A 7}$$

and simplifying the result with the aid of Parseval's theorem, we obtain (4.2)–(4.4). Invoking the above variational principle for the individual G_{mn}, we find that the approximation to G_n provided by (4.2)–(4.4) is stationary with respect to first-order variations of $g^*(y)$ about the true solution to (3.8a).

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